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**A BAYESIAN APPROACH  
TO DEMAND ESTIMATION  
AND INVENTORY PROVISIONING**

**George F. Brown, Jr.  
Warren F. Rogers, Cdr., U.S.N.**

**Research Contribution 214**

**Center  
for  
Naval  
Analyses**

**Institute of Naval Studies**

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Institute of Naval Studies  
Research Contribution 214

A BAYESIAN APPROACH TO DEMAND  
ESTIMATION AND INVENTORY PROVISIONING

July 1972

George F. Brown, Jr.  
Warren F. Rogers, Cdr., U.S.N.

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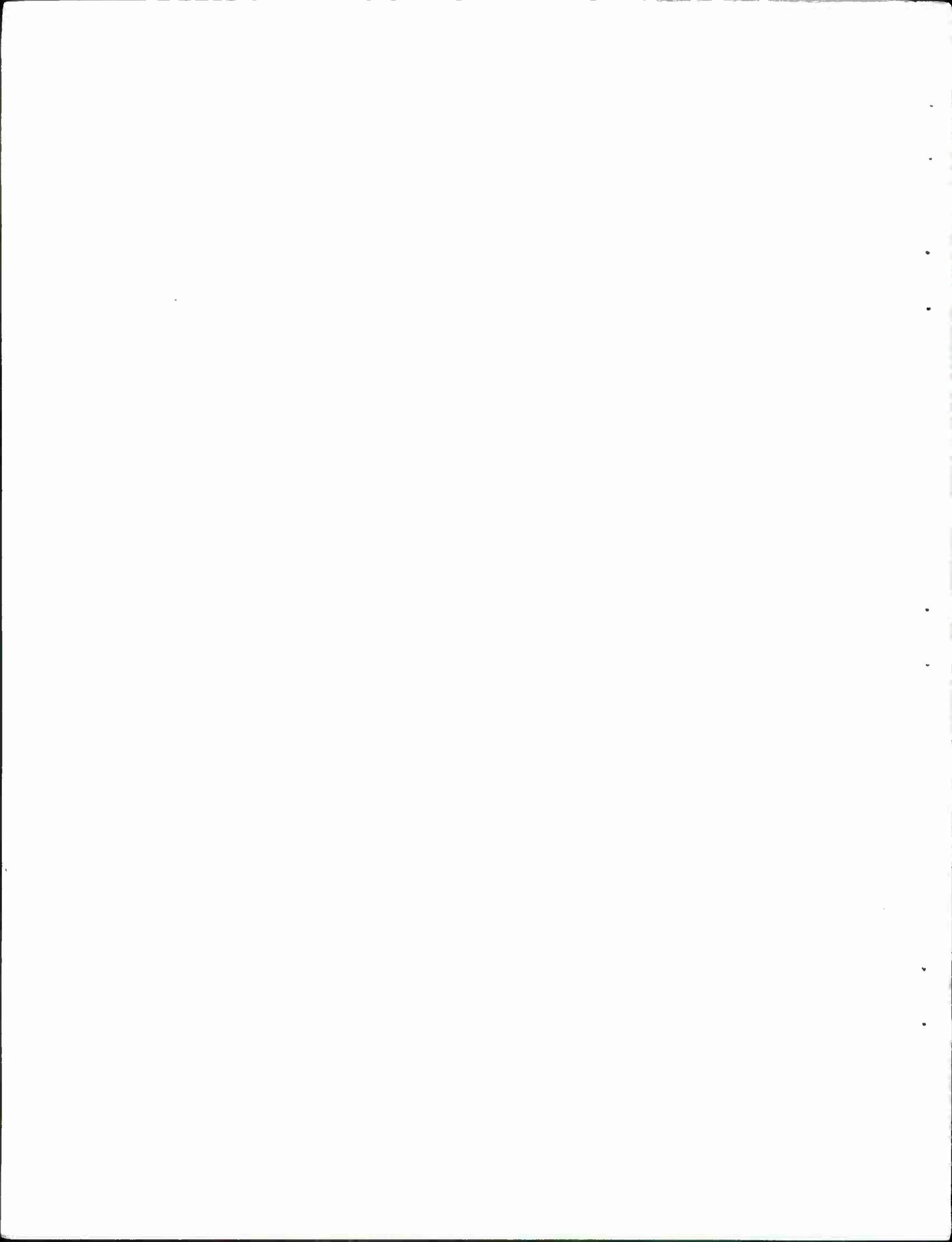
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## ABSTRACT

This Research Contribution addresses the problem of explicitly taking into account uncertainty about the demand for spare parts in making inventory procurement and stockage decisions. The model described provides for a unified treatment of the closely related problems of statistical estimation of demand and resource allocation within the inventory system, and leads to an easily implemented, efficient method of determining requirements for spare parts both in the early provisioning phase and in later periods of operations when demand data has accumulated.

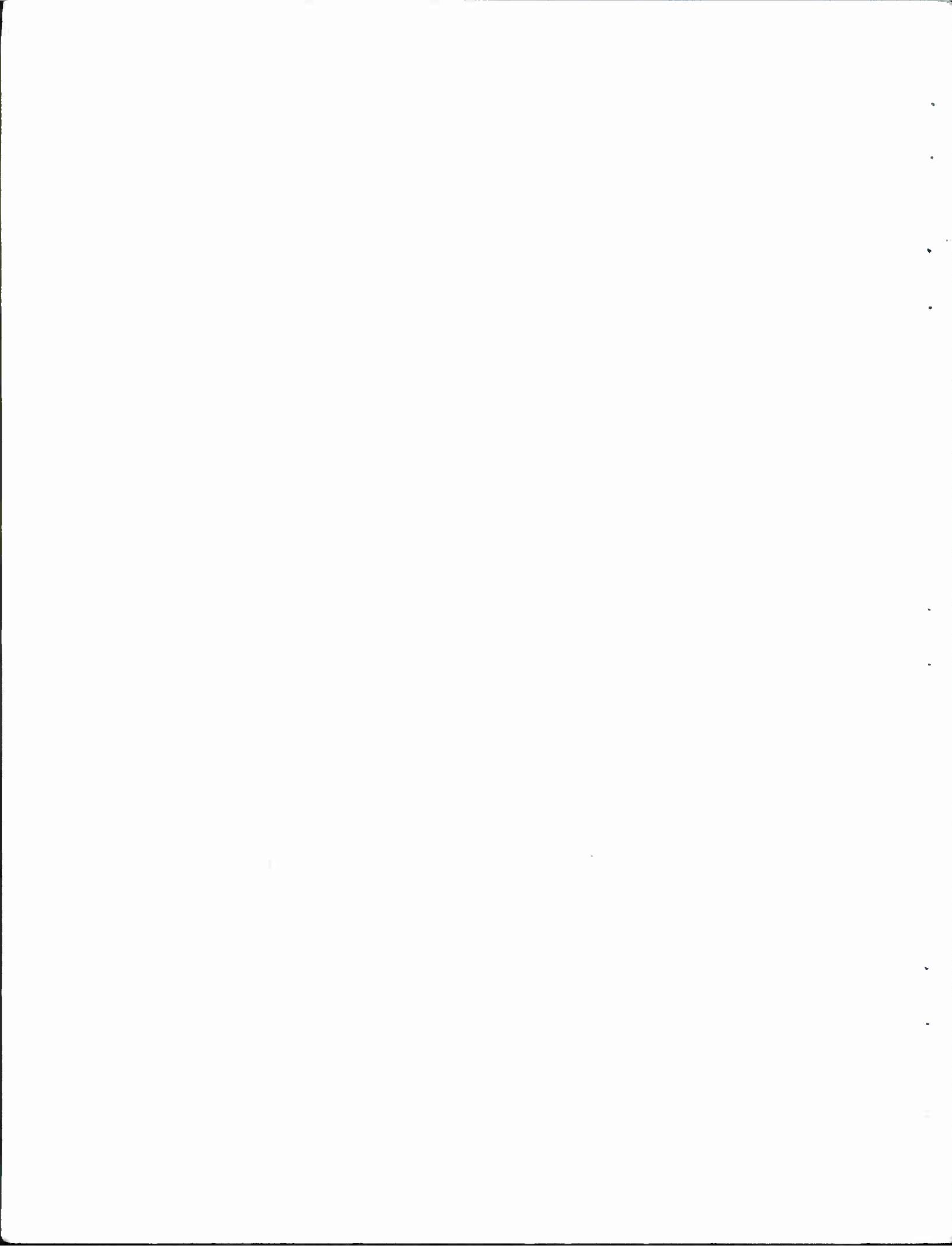
Analyses of the model's theoretical foundations and of sample outcomes of the model based upon data on parts intended for use in the F-14 lead to conclusions of great importance to both support planners and operations planners.

Finally, of particular significance is the ability afforded the planner by this model to quantify the impact on inventory system costs of varying levels of system reliability or management uncertainty as to projected system performance. This will provide an economic basis for analysis of such alternatives as early deployment, operational testing, and equipment redesign.



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## I. INTRODUCTION AND SUMMARY

Numerous studies<sup>1</sup> have demonstrated that the demand for aircraft spare parts is typically uncorrelated with identifiable program factors. In the absence of such deterministic predictors, statistical estimation procedures provide the best alternative means of estimating future requirements. Statistical estimation consists of specifying the probability distribution of demand which, in some sense, best explains the available data or, in the absence of data, best reflects the prior beliefs of the designer and the experience of the inventory manager. Having specified the probability distribution, it is necessary to determine the optimal inventory level as a function of the associated costs and budget constraints.

Typically the related problems of estimation and resource allocation are treated separately.<sup>2</sup> In simple inventory problems this is probably justified. However, when planning support of an extremely complex weapons system, with very great numbers of parts of widely varying cost and uncertain performance, a unified treatment of these problems is essential. In short, the objective must be to specify the optimal inventory decision when system performance may be projected with only limited assurance.

This paper describes such a procedure for determining optimal inventory levels for aircraft spare parts. The procedure may be used before demand data has been generated by incorporating estimates developed at provisioning and provides for progressive updating of estimates as data becomes available. The model is simple to apply and extremely efficient and requires only existing data sources. It is based on a few intuitive assumptions which have been repeatedly demonstrated to correspond closely to data on existing systems. The model may therefore be used with confidence, not only to determine inventory requirements, but perhaps more importantly to evaluate budgetary and operational implications of support policies.

An application of these procedures to a number of parts currently being provisioned for the F-14 is described. From this application and from theoretical consideration, a number of very important results are derived. The most important of these is that a spare parts inventory adequate to assure high system reliability early in system life will be very costly and extremely wasteful. The inventory required will be large but very little of it will actually be used. As data accumulates, however, it will be possible to design inventories to provide equal reliability assurance at greatly reduced cost. Thus,

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<sup>1</sup> See for example Denicoff, M., and Haber, S., "A Study of Usage and Program Relationships for Aviation Repair Parts," The George Washington University, Logistics Research Project, Serial T-140/62, 7 August 1962. The probability model we developed here is proposed in this reference and many others on empirical grounds. The fact that small correlations may be expected from data realized from this process does not appear to have been noted before.

<sup>2</sup> An exception is Zacks, S., "A Two-Echelon Multi-Station Inventory Model for Navy Applications," The George Washington University, Logistics Research Project, Technical Memorandum, Serial TM-15175, 31 July 1968. Zacks' approach is also Bayesian and uses the same probability model as that developed here.

unless there are vital, overriding operational requirements, the most desirable course of action is to accept low reliability in the early life of the system, procuring parts as needed until sufficient data has been accumulated to permit more economical inventory design.

Further significant results are summarized in the following paragraphs and are discussed in the remaining sections of the report.

1. The model described in this paper provides for a unified treatment of the closely related problems of statistical estimation of demand and resource allocation within the inventory system, which are typically treated separately. A frequent criticism of theoretical inventory models is that they do not reflect the uncertainty about the parameters which are inputs to the model -- in particular, the probability distribution of demand. The procedure described here explicitly introduces such uncertainties into the inventory decision process.

2. Uncertainty about demand distributions can result from a number of factors. At the time of initial provisioning, estimates may be quite tentative due to the lack of any operational data on which to base them. Further, a system designed to operate worldwide, in a host of unpredictable environments, with a variety of maintenance procedures and skill levels supporting it, being employed in widely varying missions, can be expected to have not one, but many, rates of demand. Both forms of uncertainty are relevant to the inventory decision and are incorporated into the model described in this report. Furthermore, these two types of uncertainty imply different requirements for inventory support.

3. The model developed in this report enables the inventory manager to incorporate all of his particular knowledge about a deployment into the optimal decision process. Peculiarities about a particular deployment or a squadron's maintenance practices, as well as the size of the squadron and the projected flying hour program, can be reflected in the inputs to the model.

4. The effect of uncertainty about the demand rate is to increase the variance of the probability distribution of demands. In turn, this high variance typically implies higher required levels of stockage, more frequent re-ordering, and, in general, higher costs of supporting the weapons system. This high variance and associated high support cost have been frequently reported in studies of Naval inventory systems. However, little guidance has been provided about what the Navy can do about these problems. Our model suggests a number of management procedures which can be employed to solve these problems beyond the usual suggestion that the equipment be redesigned so as to be made more reliable. In fact, we demonstrate, in some cases, that a reduction in uncertainty can be of more value than an equivalent increase in reliability. First, extensive operational testing can be undertaken to gather data which will lead to more certainty about demand rates. Planning to extensively deploy an untested weapons system and to support it for wartime usage will require high levels of inventory support. Furthermore, across parts, the higher the level of uncertainty, the greater will be the percentage of this inventory which will go unused. However, it is impossible, *a priori*, to tell exactly which parts will be used, so that extensive support across all such parts is required. Secondly, greater standardization of maintenance facilities and practices will reduce the variance in this demand and thus lead to lower inventory system costs. Finally, the ability of the

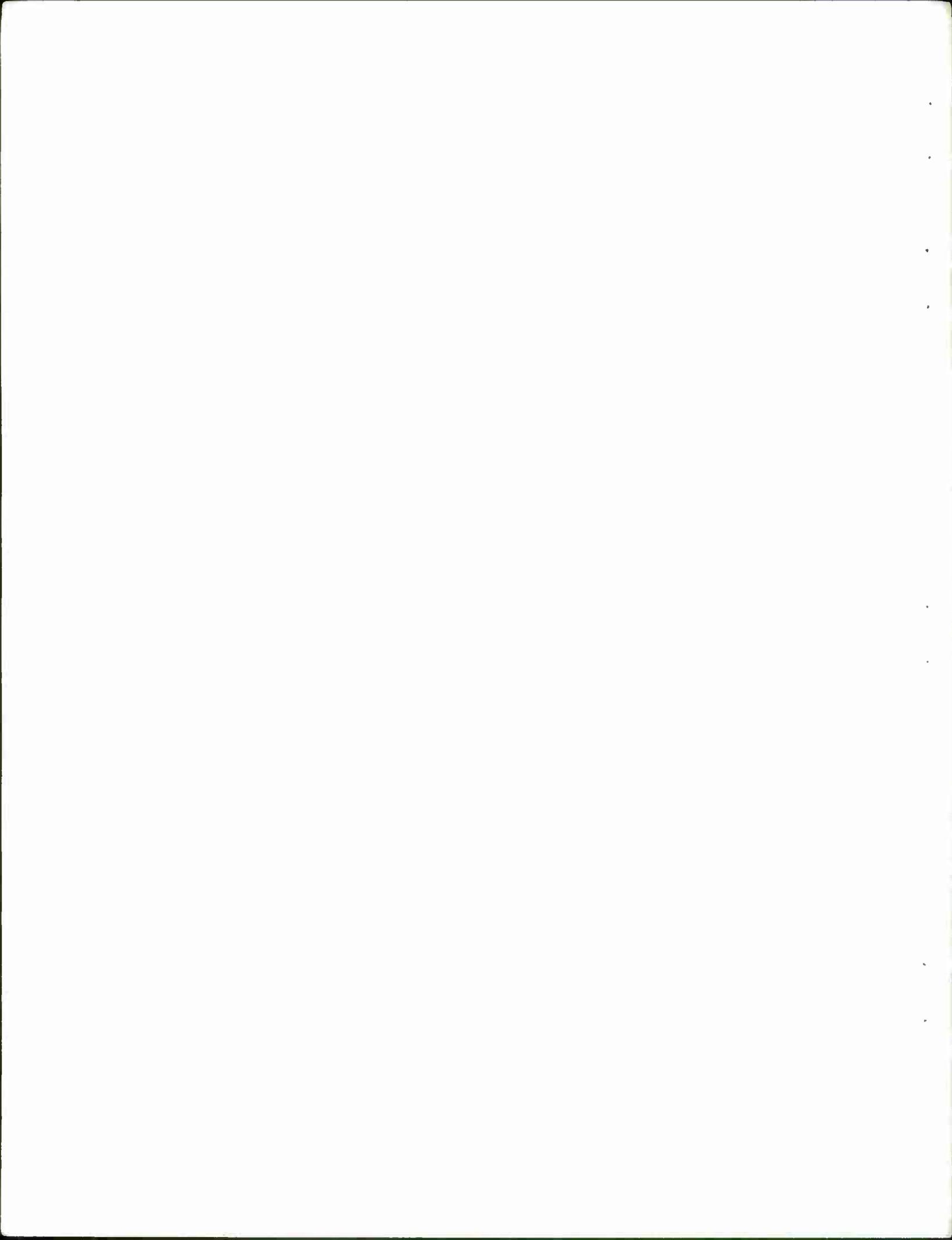
inventory system manager to incorporate information peculiar to a particular squadron and deployment can reduce the variance in demand that the inventory system must protect against.

5. Numerous empirical studies of demand data have concluded that the observed pattern of demands over time correspond well with the realizations of a compound Poisson process. The explanations advanced to support this conjecture have largely been unsatisfying. The model developed in this paper, which follows from a few relatively mild assumptions, leads to one member of the compound Poisson family -- the Negative Binomial distribution. Thus the results of this paper are supported by a wide body of previous empirical research.

6. A second major conclusion of previous empirical research has been that, with few exceptions, demands for spare parts are uncorrelated with program factors such as flight hours. The model developed in this report suggests that flight hours do enter into the determination of spare parts demands, but in a very complex and distinctly non-linear way. We show that, in fact, the theoretical model developed here predicts the finding of a lack of correlation between flight hours and demand. The optimal inventory decisions generated in the model involve a highly complex interaction among the parameters of the demand distribution, relevant costs, and flight hours. Predictions of demand based upon simple linear relations between demands and flight hours are overly naive and are based upon a faulty premise.

Many of the mathematical results in this paper are well known. They are reproduced here both for completeness and because their implications for support policy are extremely important and have not been fully explored in the past.

The implementation of the procedures described in this paper should present little difficulty to managers of the Navy's inventory systems. All of the procedures employed in the analysis, including those for determining optimal inventory decisions and for incorporating new demand information as it becomes available, have been programmed and require only a few seconds of processing time. The decision rules have been shown to be of a particularly simple form and thus can be used by managers of deployed squadrons. The Center for Naval Analyses will provide assistance in adapting the existing computer programs to other facilities.



## II. A MODEL OF SPARE PARTS DEMAND

### THE PROBABILITY MODEL

Inventory decisions in Navy Supply are typically based on point estimates of demand. When demands are subject to random variation, procedures based on point estimates will typically lead to poor decisions. An optimal inventory decision model must consider the full range of possible realizations of the random process which generates demands and their associated probabilities. The inventory model described in section IV does so.<sup>3</sup> In this section, we derive a probability model of demands which coincides well with empirical studies of demand data and is suitable for input in the inventory model.

Numerous empirical studies of demand data have been conducted.<sup>4</sup> Three conclusions emerge:

- a. With very few exceptions, demands for spare parts are uncorrelated with program factors such as flying hours.
- b. The Poisson distribution provides an adequate description of demands for parts exhibiting low demand rates.
- c. The variance of demands for high usage parts over time is typically very large compared to their mean.

The latter observation has led to rejection of a simple Poisson model of the demand process for high usage rate parts since the Poisson distribution has identical mean and variance.

Several conjectures have been offered to explain this behavior and to justify the choice of one member of the compound Poisson family of distributions.<sup>5</sup> We have found these explanations unsatisfying either because they fail to correspond to operational experience or because the models they were advanced to support would be inappropriate

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<sup>3</sup> A further treatment of the theoretical basis for this model is contained in Brown, George F. Jr., Corcoran, T. M., and Lloyd, R. M., "Inventory Models with Forecasting and Dependent Demand," Management Science, March, 1971, and "A Dynamic Inventory Model with Delivery Lag and Repair," Center for Naval Analyses, Professional Paper 3, 1969.

<sup>4</sup> For example, see Fawcett, W. M., and Gilbert, R. D., "Characteristics of Demand Distributions for Aircraft Spare Parts," General Dynamics Fort Worth Division Report ERR-FW-512, November 1966. Also see Youngs, J. W. T., Geisler, M. A., and Brown, B. B., "The Prediction of Demand for Aircraft Spare Parts Using the Method of Conditional Probabilities," RAND Corporation Report RM-1413, January 1955.

<sup>5</sup> For example, see Feeney, G. J., and Sherbrooke, C. C., "The (s-1, S) Inventory Policy under Compound Poisson Demand," RAND Corporation Memorandum RM-4176-PR, March 1966. The authors offer four conjectures to explain the high variability observed for recoverable item demand.

if in fact the explanations were valid. Instead we show that one member of the compound Poisson family, the Negative Binomial distribution, follows logically from some rather mild assumptions and some practical constraints imposed by the nature of the estimation problem.

First we assume that demands for parts in non-overlapping time intervals are statistically independent. It is easily shown (cf. Feller [c]) that any distribution on the integers which satisfies this assumption is a member of the compound Poisson family.

Next we will assume that we may describe the uncertainty which exists about the anticipated rate of failures,  $\lambda$ , by assigning to it a probability distribution which summarizes designers', manufacturers', and support managers' best "guesses" as to the values of mean time between failures which may be realized when the equipment in question is placed in operation. The treatment of demand rate as a random variable may at first appear strange to those unacquainted with Bayesian methods. Justification of this procedure is treated extensively in Raiffa and Schlaifer [d] and DeGroot [b]. In this particular application, however, it is intuitive that the underlying mean rate of failures which will be experienced when an equipment is employed in the fleet should be expected to vary randomly with varying and unpredictable environments. We will refer to the distribution of  $\lambda$  as the prior distribution.

For any given realization of failure rate per unit of time, say  $\lambda$ , we will assume that the probability of observing more than one demand in any very small increment of time is itself vanishingly small.

With this last assumption and the assumption of independence between non-overlapping time intervals we may conclude<sup>6</sup> that the conditional probability of observing  $k$  failures in any time increment  $t$  given that the rate  $\lambda$  holds is given by:

$$P[k | \lambda] = \frac{(\lambda t)^k e^{-\lambda t}}{k!} . \quad (1)$$

If we denote our prior distribution on  $\lambda$  by  $F(\lambda)$ , then the unconditional probability of observing  $k$  failures in time  $t$  is:

$$\begin{aligned} P(k) &= \int_0^{\infty} P(k | \lambda) dF(\lambda) \\ &= \int_0^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} dF(\lambda) . \end{aligned} \quad (2)$$

---

<sup>6</sup>For a rigorous statement of the postulates leading to this distribution, see Feller [c].

To this point, we have considered estimates of the distribution of demands based solely on prior considerations; that is, before demand data has been generated. Naturally, as demand data accumulates, we would wish to modify our prior beliefs about the mean demand rate to reflect this additional information. This is accomplished by an application of Bayes rule as follows. Let  $f(\lambda) = \frac{dF(\lambda)}{d\lambda}$  be the prior density of  $\lambda$  and suppose that in each of  $n$  time periods  $t_i$ , we have observed  $x_i$  demands, where  $i=1, 2, \dots, n$ . Then the conditional density of  $\lambda$ , given the observations, is:

$$f(\lambda | x_1, \dots, x_n) = \frac{\prod_{i=1}^n \pi P[x_i | \lambda t_i] \cdot f(\lambda)}{\int_0^\infty \prod_{i=1}^n \pi P[x_i | y t_i] \cdot f(y) dy} \quad (3)$$

We will refer to the conditional distribution of  $\lambda$  given the observations as the posterior distribution of  $\lambda$ .

With the additional information about demand rate summarized by the posterior distribution, the unconditional distribution of demands in equation (2) now becomes:

$$P[k] = \int_0^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!} f(\lambda | x_1, \dots, x_n) d\lambda \quad (4)$$

#### CHOOSING THE PRIOR DISTRIBUTION

To determine a suitable prior distribution  $F(\lambda)$ , Raiffa and Schlaifer [d] establish the following desiderata:

- "1.  $F$  should be analytically tractable in three respects: (a) it should be reasonably easy to determine the posterior distribution resulting from a given prior distribution and a given sample; (b) it should be possible to express in convenient form the expectations of some simple utility functions with respect to any member of  $F$ ; (c)  $F$  should be closed in the sense that if the prior is a member of  $F$ , the posterior will also be a member of  $F$ .
- 2.  $F$  should be rich, so that there will exist a member of  $F$  capable of expressing the decision maker's prior information and beliefs.
- 3.  $F$  should be parametrizable in a manner which can be readily interpreted, so that it will be easy to verify that the chosen member of the family is really in close agreement with the decision maker's prior judgments about  $\theta$  and not a mere artifact agreeing with one or two quantitative summarizations of these judgments."

It is of particular importance in this application that the criterion 1(c) apply. If we chose a prior distribution for which it did not, then the posterior distribution realized after each period of data collection would have an algebraic form differing from that of the preceding stage. Thus, extensive reprogramming would be required at each stage, effectively limiting the practical usefulness of the procedure. We therefore choose a family of distributions which satisfies 1(c) and examine its other properties.

A random variable  $\lambda$  is said to be distributed as the two parameter Gamma distribution with parameters  $\alpha$  and  $\beta$ , denoted  $G_{\alpha, \beta}$ , if its density is:

$$f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} . \quad (5)$$

If the parameter  $\lambda$  in the Poisson density given in equation (5) has a prior distribution,  $G_{\alpha, \beta}$ , and if  $x_i$ ,  $i=1, 2, \dots, n$ , are  $n$  independent samples from that Poisson process, then the posterior distribution of  $\lambda$  is again a Gamma distribution with revised parameters

$$\alpha' = \alpha + \sum_{i=1}^n x_i, \quad \beta' = \beta + n .$$

Thus a Gamma prior satisfies criterion 1(c) and coincidentally 1(a).

For this application the utility function is defined implicitly by the inventory program and thus criterion 1(b) reduces to the requirement that the unconditional distribution of demands be computationally tractable. From equations (2) and (5) we derive the unconditional distribution of demands as:

$$\begin{aligned} P(k) &= \int_0^\infty \frac{\lambda^k e^{-\lambda}}{k!} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} d\lambda \\ &= \binom{\alpha+k-1}{k} \left(\frac{\beta}{\beta+1}\right)^\alpha \left(\frac{1}{\beta+1}\right)^k , \end{aligned} \quad (6)$$

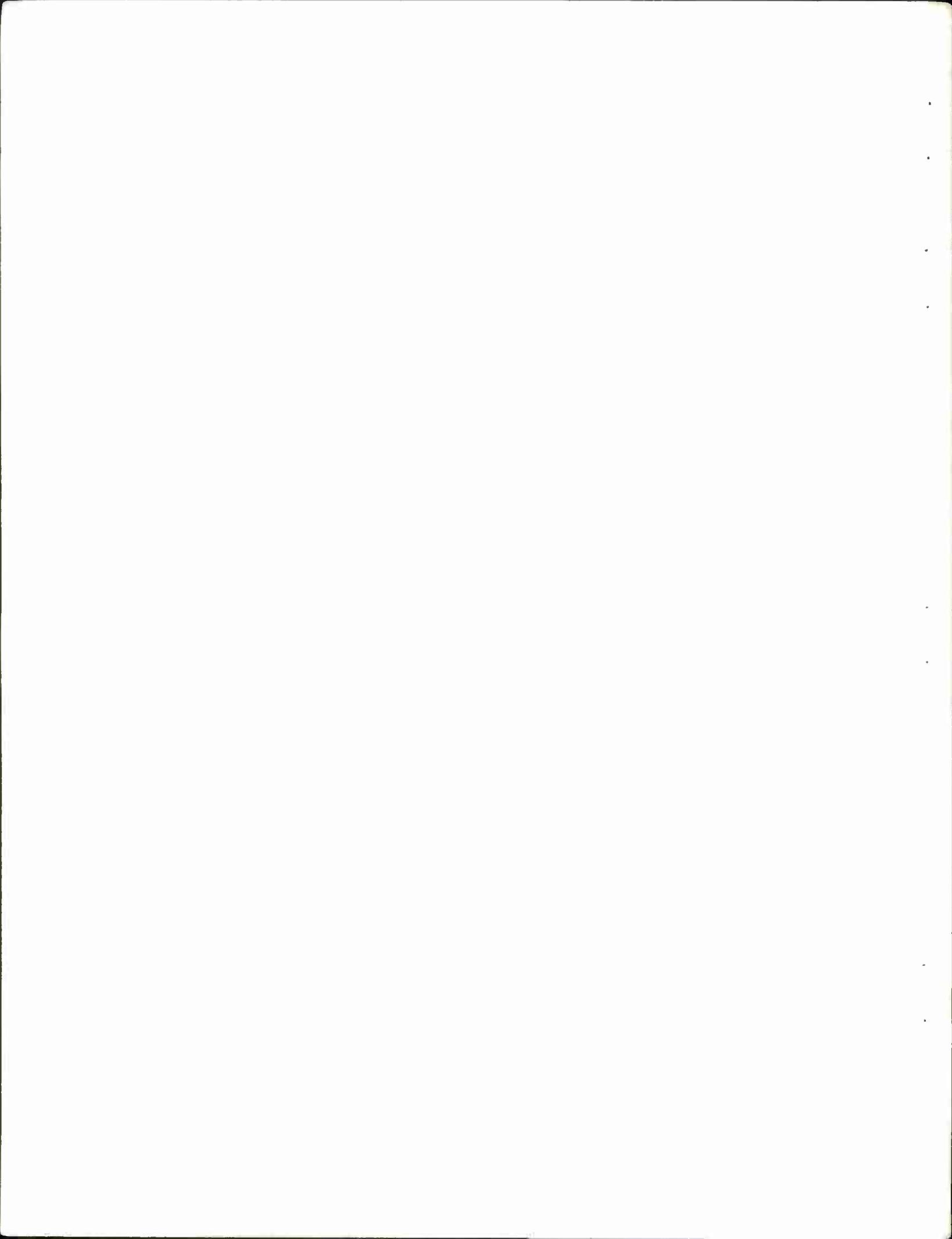
the Negative Binomial distribution with parameters  $\alpha$  and  $\frac{\beta}{\beta+1}$ . A simple recurrence relation which simplifies computation of the probabilities is given in section IV.

The Gamma family provides an extremely wide range of shapes, amply satisfying the second major criterion.

The final criterion deserves more extended consideration. The Gamma distribution is completely characterized by its mean and variance or by the mode and variance. The expected value (the mean) and the most likely value (the mode) of the rate of demands are probably meaningful concepts to an inventory manager or provisioner. It is doubtful, however, that variance is an equally meaningful concept and that prior estimates of it would really reflect their prior beliefs as to likely system performance. This question is treated in more detail in INS Study 37 [e].

It is of interest to note that while the distribution in equation (6) is compound Poisson, the random process over the time parameter  $t$  is not. In fact, a non-degenerate mixture of Poisson processes cannot yield a compound Poisson process. The distribution in equation (5) is in fact infinitely divisible in the parameter  $\alpha$  not in  $t$ .<sup>7</sup>

<sup>7</sup>We are grateful to Dr. Joseph Bram, who drew this point to our attention.



### III. SOME IMPORTANT IMPLICATIONS OF THE PROBABILITY MODEL DEPLOYMENT OF NEW WEAPONS SYSTEMS

The model we have described has great intuitive appeal in that its development follows from a relatively few, mild assumptions, all of which appear consistent with operational experience. In addition there is strong (and plentiful) empirical evidence that the model accurately reflects real world experience. Predictions based on this model therefore merit serious consideration, particularly in view of their implications for wartime contingency planning.

The demand distribution given in equation (6) has mean and variance

$$E(k) = \frac{\alpha t}{\beta} ,$$

$$\text{Var}(k) = \frac{\alpha t(\beta+t)}{\beta^2} .$$

The mean and variance of the prior distribution given in equation (5) are:

$$E(\lambda) = \frac{\alpha}{\beta} ,$$

$$\text{Var}(\lambda) = \frac{\alpha}{\beta^2} .$$

A large prior variance which implies a large uncertainty about  $\lambda$  is thus reflected in a large unconditional variance of demand. In addition the variance of the demand distribution increases quadratically in flying hours  $t$ . The immediate implication of increasing variance is that the probabilities of large demands also increase. To provide desired system reliability it is then necessary to procure larger inventories. But for fixed mean demand an increase in demand variance also implies an increase in the probability that no demands will in fact occur.<sup>8</sup> Thus the likelihood that expenditures will be wasted also becomes large. Of course across parts, it is impossible to tell with certainty which will be required and which will not.

Variance of demand is controlled by several factors. First there is the reliability of the system, reflected in the prior mean,  $\frac{\alpha}{\beta}$ . Then there is the variance of the prior,  $\frac{\alpha}{\beta^2}$ , which reflects the state of uncertainty about the current estimate of demand rate. Finally, there is the projected flying hour rate.

<sup>8</sup>The mean demand per flying hour is always less than one, so that for a fixed mean the increased mass at large values must be "balanced" by an increase at zero.

One conclusion is immediate. A new weapons system, incorporating "state of the art" equipment, whose performance may be projected only with great uncertainty, will require a large inventory of spare parts to ensure acceptable reliability. If, in addition, it is intended that the system be capable of sustaining an intensive wartime flying program, then the inventory must be expanded many times over. In fact, the sample calculations given in section IV indicate that even with the penalty cost fixed at the peacetime rate, which is no doubt unrealistically low, the war reserve inventory necessary to ensure high reliability in the absence of resupply would far exceed the levels normally maintained.

An inventory policy designed to provide for wartime employment early in the life of the system would not only be costly but also extremely wasteful. It is important to realize the distinction between the planned inventory necessary to assure readiness and the usage which will actually be generated by the random process used in planning. The inventory must be designed to guard against demands whenever there is significant probability that they will occur. The demand actually realized will reflect the fact that there is also significant probability that a specific part will experience few or even zero demands.

The alternative is to defer some procurement decisions until the acquisition of demand data permits more reliable prediction of demand rates. As noted in section II, the posterior Gamma distribution of demand rate after  $n$  realizations of the process yielding demands  $X_i, i=1, 2, \dots, n$ , has parameters:

$$\alpha + \sum_{i=1}^n X_i, \beta + n .$$

Then the posterior unconditional distribution of demands has variance:

$$\text{Var}(k) = \frac{(\alpha + \sum_{i=1}^n X_i) t (\beta + n + t)}{(\beta + n)^2}$$

and thus the posterior variance decreases roughly as  $\frac{1}{n}$ . It follows that, in addition to allowing management to isolate those equipments whose realized reliability will dictate redesign action, deferral of major commitment of resources enables us to design future inventories providing the desired level of readiness assurance but at a greatly reduced cost.

Deferral of procurement, of course, implies acceptance of a reduced state of readiness in the early stages of the program so that enhanced readiness and the ability to respond to contingencies can be realized in later stages at acceptable cost. If, however, an initial high state of readiness and ability to respond to contingencies is deemed imperative, then the inventory should be planned realistically in the full realization that it will involve very great cost and potential waste.

## DEMAND TO FLYING HOUR CORRELATION

It has been noted earlier that estimates of the correlation between demand (or failures) and flying hours based on observed data are typically small. We now demonstrate that this should in fact be the expected outcome from data generated by our probability model.

Treating flying hours  $t$  as a random variable, we calculate the population correlation between  $t$  and  $k$ , the number of failures, as follows.

The covariance of  $t$  and  $k$  may be written

$$\begin{aligned}\text{Cov}(t, k) &= E(t(k - E(k))) \\ &= E(tk) - E(t)E(k) .\end{aligned}$$

$$\begin{aligned}\text{Then } E(k) &= E(E(k) | t) \\ &= E\left(\frac{\alpha}{\beta} t\right) \\ &= \frac{\alpha}{\beta} E(t) ,\end{aligned}$$

$$\begin{aligned}E(tk) &= E(E(tk | t)) \\ &= E\left(\frac{\alpha}{\beta} t^2\right) \\ &= \frac{\alpha}{\beta} E(t^2) ,\end{aligned}$$

$$\text{whence } \text{Cov}(t, k) = \frac{\alpha}{\beta} \text{Var}(t) .$$

$$\text{Since } \text{Var}(k) = E(k^2) - E^2(k)$$

$$\begin{aligned}\text{and } E(k^2) &= E(Ek^2 | t) \\ &= E\left(\frac{\alpha t(\beta+t)}{\beta^2} + \left(\frac{\alpha}{\beta} t\right)^2\right) \\ &= \frac{1}{\beta^2} (E(\alpha\beta t) + E(\alpha t^2) + \alpha^2 E(t^2)) ,\end{aligned}$$

$$\begin{aligned}\text{Var}(k) &= \frac{\alpha}{\beta} E(t) + \frac{\alpha}{\beta^2} E(t^2) + \frac{\alpha^2}{\beta^2} E(t^2) - \frac{\alpha^2}{\beta^2} E^2(t) \\ &= \frac{\alpha}{\beta} (E(t) + \frac{1}{\beta} E(t^2) + \frac{\alpha}{\beta} \text{Var}(t)) .\end{aligned}$$

Thus

$$\begin{aligned}\text{Corr}(t, k) &= \left( \frac{\frac{\alpha}{\beta} \text{Var}(t)}{E(t) + \frac{1}{\beta} E(t^2) + \frac{\alpha}{\beta} \text{Var}(t)} \right)^{1/2} \\ &= \left( \frac{1}{1 + \frac{\beta}{\alpha} \frac{E(t)}{\text{Var}(t)} + \frac{1}{\alpha} \frac{E(t^2)}{\text{Var}(t)}} \right)^{1/2} \\ &< \left( \frac{1}{1 + \frac{\beta}{\alpha} \frac{E(t)}{\text{Var}(t)}} \right)^{1/2}\end{aligned}$$

Now  $\frac{\alpha}{\beta}$  is the expected number of demands per flying hour, which is typically very small, so that strong correlation will exist only if the variance of  $t$  is large relative to its mean. We are thus led to the somewhat vacuous conclusion that correlations will be large only if flying hours are extremely variable and thus cannot be predicted with assurance.

From our earlier discussion of the probability model, it should be apparent that demands are not statistically independent of flying hours, but it should also be clear that the dependence is distinctly non-linear. The optimal inventory decisions generated in the model involve highly complex interactions among the parameters of the distribution, the relevant costs, and flying hours. Predictions of demand based on simple linear relations between demands and flying hours are overly naive and, as the discussion here shows, are based on a faulty premise.

#### IV. AN APPLICATION TO INVENTORY MANAGEMENT

##### CALCULATION OF PROBABILITIES REQUIRED FOR THE INVENTORY MODEL

The inventory model employed in this analysis employs dynamic programming techniques to determine the optimal order size in each period,  $y_t$ , and the optimal initial stockage,  $I_0$ , using a single state variable,  $J_t$ , the number of items on hand, on order (but not yet delivered), and in repair at the end of period  $t$ . Defining  $f_t(J)$  as the total discounted expected costs under control of the inventory manager from period  $t$  to the end of the planning horizon, given  $J$  units on hand, on order, and in repair, following an optimal policy, the following recursive relationship may be used to determine  $y_t$  and  $I_0$ :

$$f_t(J_{t-1}) = \begin{cases} 0 & \text{for } t = T - \ell_1 + 1, \dots, T+1 \\ \min_{y_t \geq 0} \{ K\delta(y_t) + \alpha E f_{t+1}[J_{t-1} + y_t - D_t + R_t] \\ \quad + \alpha^{\ell_1} G_{t+\ell_1}(J_{t-1} + y_t) \} & \text{for } t = 1, \dots, T - \ell_1 \\ \alpha f_1(I_0) + G'(I_0) + K\delta(I_0) & \text{for } t = 0 \end{cases}$$

where

$G_{t+\ell_1}(J)$  = expected holding and penalty costs during period  $t+\ell_1$ , given  $J$  units on hand, on order, and in repair at the beginning of period  $t$  ;  
 $G'(I_0)$  = expected holding and penalty costs during periods  $1, 2, \dots, \ell_1$ , plus initial holding costs, given a starting on hand inventory of  $I_0$  .

This inventory model is designed to be used with any distribution of demand. Three probability calculations are required:

1. Probability of  $k_1$  failures in  $n$  decision periods.
2. Probability of  $k_2$  non-repairable failures in  $m$  decision periods, given a probability  $p$  that a failed part is repairable.
3. The probability that in two non-overlapping time intervals,  $t_1$  and  $t_2$  of length  $n$  and  $m$  periods respectively, a total of  $k$  failures and non-repairable failures will be observed where all failures are recorded in  $t_1$  and only non-repairable failures are recorded in  $t_2$  .

The first calculation follows immediately from equation (6).

If  $t$  is the number of flight hours per aircraft per period and the distribution of  $\lambda$ , the rate of failures, is  $G_{\alpha, \beta}$ , then

$P[k \text{ failures in one aircraft in one period}]$

$$\begin{aligned} &= \int_0^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta \lambda} d\lambda \\ &= \binom{\alpha+k-1}{k} \left( \frac{\beta}{\beta+t} \right)^\alpha \left( \frac{t}{\beta+t} \right)^k . \end{aligned} \quad (7)$$

Then if  $r$  is the number of aircraft,

$$\begin{aligned} P[k_1 \text{ failures in } n \text{ periods}] &\stackrel{\text{def}}{=} P_{n,r}(k_1) \\ &= \binom{n r \alpha + k_1 - 1}{k_1} \left( \frac{\beta}{\beta+t} \right)^{n r \alpha} \left( \frac{t}{\beta+t} \right)^{k_1} . \end{aligned} \quad (8)$$

This result follows because the Negative Binomial is reproductive with respect to  $\alpha$ . Note that we are modeling the  $n$  period,  $r$  aircraft process as the sum of  $n \cdot r$  independent replications of the basic process in equation (7). An alternative formulation would result if we considered a single process and  $nrt$  flying hours as follows:

$$\begin{aligned} P_{nr}(k) &= \int_0^\infty \frac{(\lambda nrt)^k e^{-\lambda nrt}}{k!} dF(\lambda) \\ &= \binom{\alpha+k-1}{k} \left( \frac{\beta}{\beta+nrt} \right)^\alpha \left( \frac{nrt}{\beta+nrt} \right)^k . \end{aligned} \quad (9)$$

However, the intent here is to incorporate the uncertainty about the value of  $\lambda$  which arises in large part from the variability and unpredictability of the environments in which individual aircraft will operate at different times, and thus the representation in equation (8) is appropriate.

For the second calculation we require the following.

Theorem. If failures are distributed as the Negative Binomial with parameters  $\alpha$  and  $(\frac{\beta}{\beta+t})$  and the probability that a failed part is repairable is  $p$ , then the distribution of non-repairable failures is again Negative Binomial with parameters  $\alpha$  and  $(\frac{\beta}{\beta+(1-p)t})$ .

Proof.

$P [k \text{ non-repairable failures}]$

$$\begin{aligned}
&= \sum_{x=k}^{\infty} P [k \text{ non-repairable failures} \mid x \text{ failures}] . \\
&\quad \cdot \binom{\alpha+x-1}{x} \left(\frac{\beta}{\beta+t}\right)^{\alpha} \left(\frac{t}{\beta+t}\right)^x \\
&= \sum_{x=k}^{\infty} \binom{x}{k} p^{x-k} (1-p)^k \binom{\alpha+x-1}{x} \cdot \left(\frac{\beta}{\beta+t}\right)^{\alpha} \left(\frac{t}{\beta+t}\right)^x \\
&= \left(\frac{(1-p)t}{\beta+t}\right)^k \left(\frac{\beta}{\beta+t}\right)^{\alpha} \frac{1}{k!} \sum_{x=0}^{\infty} \frac{(\alpha+x+k-1)!}{x! (\alpha-1)!} \cdot \left(\frac{pt}{\beta+t}\right)^x \\
&= \left(\frac{(1-p)t}{\beta+t}\right)^k \left(\frac{\beta}{\beta+t}\right)^{\alpha} \binom{\alpha+k-1}{k} \cdot \sum_{x=0}^{\infty} \binom{\alpha+x+k-1}{x} \left(\frac{pt}{\beta+t}\right)^x \\
&= \left(\frac{(1-p)t}{\beta+t}\right)^k \left(\frac{\beta}{\beta+t}\right)^{\alpha} \binom{\alpha+k-1}{k} \left(1 - \frac{pt}{\beta+t}\right)^{-(\alpha+k)} \\
&= \binom{\alpha+k-1}{k} \left(\frac{\beta}{\beta+(1-p)t}\right)^{\alpha} \left(\frac{(1-p)t}{\beta+(1-p)t}\right)^k \quad \text{Q.E.D.}
\end{aligned}$$

Again exploiting the reproductive quality of the Negative Binomial, we have the probability that  $r$  aircraft generate  $k_2$  non-repairable failures in  $m$  periods:

$$P_{m,r}(k_2) = \binom{mr\alpha + k_2 - 1}{k_2} \left(\frac{\beta}{\beta+(1-p)t}\right)^{mr\alpha} \left(\frac{(1-p)t}{\beta+(1-p)t}\right)^{k_2} . \quad (10)$$

The third calculation may now be carried out directly:

$$P[k_1 + k_2 = k] = \sum_{j=0}^k P_{n,r}(k-j) P_{m,r}(j) . \quad (11)$$

Calculations are simplified by use of the following simple recurrence. If  $K$  is distributed as the Negative Binomial with parameters  $a$  and  $b$ ,

$$\begin{aligned}
P[K = k+1] &= \frac{a+k}{k+1} (1-b) P[K = k] \\
P[K = 0] &= b^a .
\end{aligned}$$

#### EMPIRICAL RESULTS FOR F-14 PARTS

This section contains empirical results from an application of the procedures described in the preceding section to parts currently being provisioned for the F-14. While a number of the results summarized here have been predicted by the theory, they give illustrations of the great magnitude of the effects of these influences.

The first table presents results for the F-14 nose landing gear as a function of the degree of uncertainty about the failure rate.<sup>9</sup> A wide range of the parameters  $\alpha$  and  $\beta$  was chosen to illustrate prior distributions all having the same mean, but with increasing uncertainty (or variance). Each of these prior distributions implies the parameters of the unconditional demand distribution (the Negative Binomial), which are also tabulated. Finally, three outputs of the inventory model are included:

- (1) The optimal initial stockage
- (2) The optimal re-order policy,

which is of the  $(s, S)$  form. If  $X$  is the stock on hand, on order, and in repair at the beginning of a period, the optimal re-order policy is do not order if  $X \geq s$  and order  $S - X$  if  $X < s$ .

- (3) The expected inventory system costs,

over a six-month cruise, if an optimal policy is followed (for a deckload of 24 aircraft, each flying an average of one hour per day).

The first column in the table corresponds closely with a simple Poisson distribution. The mean and the variance of the unconditional demand distribution are virtually identical. This results from the fact that the variance of the prior is extremely small; failure rates different from the prior mean are felt to be very unlikely and are given little weight. Moving across each table, these results are presented for cases in which the uncertainty about the true failure rate grows larger; thus the variance in the unconditional demand distribution also increases. As the uncertainty grows, the inventory system costs and the required stockage levels also increase rapidly. These empirical results clearly demonstrate the high costs associated with uncertainty about the demand distribution, and show the importance of the management actions which can be taken to reduce this uncertainty. Early in the provisioning process, it is unlikely that there would be great confidence about the demand rate; thus, if parts are procured at this time, the high inventory system costs associated with uncertainty must be incurred. Planning for full deployment of a weapons system before much information about it is gathered could potentially require support at a cost much higher than would be required later in its service life.

The great costs associated with uncertainty are further illustrated in table II. There, inventory system costs are presented for a range of means and variances of the unconditional demand distribution. A decrease in the mean represents an increase in "reliability," while an increase in the variance represents a greater degree of uncertainty about the mean. The surprising conclusion that comes from this table is that uncertainty may be more expensive to the Navy than unreliability. Changes in the mean (holding variance constant) affect inventory system costs very little, while changes in the variance (holding the mean constant) produce much greater cost increases. Hence programs to redesign equipment may have very little impact unless greater certainty results from the redesign process.

Table III shows the effect of changes in the flying hour program on optimal stockage and re-order policies, and on expected inventory system costs. We have previously shown that, while demands cannot be predicted by means of a naive relationship with the

<sup>9</sup>Similar tables for additional parts appear in INS Study 37.

TABLE I  
OPTIMAL INVENTORY POLICIES FOR THE NOSE LANDING GEAR

$\alpha$	14	.224	.112	.056	.028	.021	.014	.0105	.007	.0035
$\beta$	1000	16	8	4	2	1.5	1	.75	.5	.25
Mean of prior distribution	.014	.014	.014	.014	.014	.014	.014	.014	.014	.014
Variance of prior distribution	.014 $\times$ $10^{-3}$	.000875	.00175	.0035	.007	.0093	.014	.0187	.028	.056
Mean of unconditional demand distribution	.014	.014	.014	.014	.014	.014	.014	.014	.014	.014
Variance of unconditional demand distribution	.014	.014875	.01575	.017	.021	.023	.028	.044	.056	.07
Optimal initial stockage	2	2	2	2	3	3	3	3	4	6
Optimal $(s, S)$ re-order policy	(1, 2)	(1, 2)	(1, 2)	(2, 3)	(2, 3)	(2, 3)	(2, 3)	(2, 3)	(3, 4)	(5, 6)
Expected inventory system costs (\$)	79,476	82,027	84,822	90,298	98,278	103,201	112,566	121,114	134,767	153,968

TABLE II  
EFFECTS OF RELIABILITY AND UNCERTAINTY  
ON EXPECTED INVENTORY SYSTEM COSTS\*

Mean Variance \	0.010	0.012	0.014	0.016	0.018
0.010	64,716				
0.012	71,418	72,003			
0.014	77,812	78,305	79,439		
0.016	84,010	84,595	85,604	87,142	
0.018	89,919	90,755	91,861	93,316	95,196

\*For nose landing gear

TABLE III  
EFFECTS OF CHANGES IN FLIGHT HOUR PROGRAM\*

Flight hours**	Optimal initial stockage	Optimal re-order policy	Expected inventory system costs
0.5	2	(1, 2)	57,685
0.75	2	(1, 2)	73,000
1.0	2	(1, 2)	90,298
1.25	3	(2, 3)	105,454
1.5	3	(2, 3)	120,837
2.0	4	(3, 4)	151,575
2.5	5	(4, 5)	180,982
3.0	6	(5, 6)	209,899
4.0	7	(7, 8)	267,278
5.0	8	(8, 9)	321,050

\*For nose landing gear,  $\alpha = 0.056$ ,  $\beta = 4.0$

\*\*Average daily flight hours per aircraft

flying hour program, the flying hour program does enter in the demand distribution in a complex way and thus must affect resource allocation decisions. These points are clearly demonstrated in the table -- higher flying hour programs require greater inventory investment and are much more expensive. Furthermore, the greater the uncertainty about the system, the greater will be the increase in this investment. Wartime flying hour programs with a system about which there is great uncertainty will require enormous inventory investments. The potential value of management actions aimed at reducing uncertainty again becomes apparent.

Finally, table IV illustrates the fact that greater uncertainty about a system also leads to greater potential wastage. Presented in the table are the probabilities of demands of various sizes on a single day for a system with mean of 0.014 and the variances listed. As the variance increases, two things happen: the probability of zero demands increases and the probability of large demands increases. Thus, while the inventory decision must provide insurance against these high demands and the associated lessening of readiness, the probability of this insurance being wasted also increases. While the changes in the probability of zero demands seem small numerically, over an extended period of time, these small changes become significant. Again, a reduction in uncertainty will lead to a decrease in both required stockage and potential wastage.

TABLE IV  
THE NEGATIVE BINOMIAL DENSITY FUNCTION

Variance	Probability that demands* on a single day will be:											
	0	1	2	3	4	5	6	7	8	9	10	11
.014	.98610	.01379	.00010	.00000	.00002	.00000						
.014875	.98651	.01300	.00047	.00006	.00001	.00000						
.01575	.98689	.01228	.00076	.00006	.00002	.00000						
.017	.98758	.01106	.00117	.00016	.00002	.00000						
.021	.98871	.00923	.00158	.00036	.00009	.00002	.00001	.00000				
.023	.98933	.00831	.00170	.00046	.00014	.00004	.00001	.00000				
.028	.99034	.00693	.00176	.00059	.00022	.00009	.00004	.00002	.00001	.00000		
.044	.99114	.00595	.00172	.00066	.00028	.00013	.00006	.00003	.00002	.00001		
.056	.99234	.00463	.00155	.00069	.00035	.00019	.00010	.00006	.00003	.00002	.00001	.00000
.070	.99438	.00278	.00112	.00060	.00036	.00023	.00015	.00011	.00007	.00005	.00004	.00003

\*For part with mean 0.014 (such as nose landing gear).

## REFERENCES

- (a) Brown, G. F., Jr., and Perlman, B. L., "Optimal Inventory Management for Naval Aviation Support," Center for Naval Analyses, Research Contribution 186, 1971
- (b) DeGroot, M. H., Optimal Statistical Decisions, McGraw-Hill Book Co., New York, 1970
- (c) Feller, W., An Introduction to Probability Theory and Its Applications, John Wiley and Sons, Inc., New York, Volume I, 1957, Volume II, 1966.
- (d) Raiffa, H. and Schlaifer, R., Applied Statistical Decision Theory, the M.I.T. Press, Cambridge, Mass., 1961.
- (e) Center for Naval Analyses, "F-14/Phoenix Provisioning: A Study of Initial Spare Part Provisioning of A New Weapons System," Institute of Naval Studies Study 37, 1972.

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